1. Introduction

Theorem 1. *Let* $\delta : (S^1)^{2^n} \to \mathbb{R}^{2^n-1}$ *be a continuous map with the w-antipodal property for each word* $w \in W$*. Suppose there exists* $\mathbf{c} \in (S^1)^{2^n}$ *such that* $\delta(\mathbf{c}) = \mathbf{0}$*. Then the zero set of* δ *contains a connected subset* Λ *which can be approximated arbitrarily closely in the Hausdorff distance by a smooth loop L, for which the projection map:* $p_0 : L \to S^1$, $p_0(y_0, y_1, \ldots) = y_0$ *is not null-homotopic.*

Proof. Let $\{\delta_m\}$ $(m = 1, 2, \ldots)$ be smooth approximations of δ such that

(a) Given $\epsilon > 0$, we have for $\mathbf{x} \in (S^1)^{2^n}$, and large m,

$$d(\delta_m(\mathbf{x}), \delta(\mathbf{x})) < \epsilon$$

(d being the standard metric on $\mathbb{R}^{2^n-1}$)

(b) $\delta_m(\mathbf{c}) = \mathbf{0}$, for a fixed $\mathbf{c}$.

(c) For each $w \in W$, $\tilde{p}_w(\delta_m(\mathbf{x})) = -\tilde{p}_w(\delta_m(\mathbf{x}))$.

Existence of smooth approximations $\{\delta_m\}$ for each $m \in \{1, 2 \ldots, \}$ that satisfy conditions (a) and (b) follows from the smooth approximation theorem (Thm 2.5, pg 8 of [3]). Existence of smooth approximation δ_m that satisfy (c) in addition to (a) and (b) be seen follows. Let $\delta_{m'}$ be a smooth approximation of δ satisfying (a) and (b). We define $\delta_m : (S^1)^{2^n} \to \mathbb{R}^{2^n-1}$ by

$$\tilde{p}_w(\delta_m(\mathbf{x})) = \tilde{p}_w((\delta_{m'}(\mathbf{x}) - \delta_{m'}(\omega_w(\mathbf{x})))/2)$$

for each $w \in W$. It is then clear that, for each $w \in W$, δ_m is a smooth map that satisfies all three conditions (a), (b) and (c).

To prove this lemma, we will show that the zero sets of the sequence of smooth approximations $\{\delta_m\}$ to δ, have closed connected subsets $\{\Lambda_m\}$ for which the projection map $p_0 : \Lambda_m \to S^1$ is not null-homotopic and find a subsequence which converges to a connected closed subset Λ of the zero set of δ.

We now construct a function $h : (S^1)^{2^n} \to \mathbb{R}^{2^n-1}$ that satisfies the w-antipodal property for each $w \in W$ and also satisfies

$$h(\mathbf{c}) = \mathbf{0}$$

for a fixed $\mathbf{c}$. Write $\mathbf{c} = (e^{ic_0}, e^{ic_1}, e^{ic_{11}}, \ldots, e^{c_{12\ldots2}}) \in (S^1)^{2^n}$. For $w \in W$, and $k \le \mathrm{len}(w)$ let w_k the subword of w of length k. We write, $\mathbf{x} = (e^{iu_0}, e^{iu_1}, e^{iu_{11}}, \ldots, e^{iu_{12\ldots2}}) \in (S^1)^{2^n}$, and define maps $g_k : W \times (S^1)^{2^n} \times (S^1)^{2^n} \to \mathbb{R}^{2^n-1}$,

$$\begin{aligned} g_k(w, \mathbf{c}, \mathbf{x}) = {} & (1 + \cos(u_{w_k} - c_{w_k}))g_{k+1}(w, \mathbf{c}, \mathbf{x})/2 \\ & - (1 - \cos(u_{w_k} - c_{w_k}))g_{k+1}(w, \omega_{w_k}(\mathbf{c}), \mathbf{x})/2 \end{aligned} \tag{1}$$

and $g_l(w, \mathbf{c}, \mathbf{x}) = \sin(u_w - c_w)$ where $l = \mathrm{len}(w)$.

We define map $h : (S^1)^{2^n} \to \mathbb{R}^{2^n-1}$ as follows: for each $w \in W$, define

$$\tilde{p}_w(h(\mathbf{x})) = g_1(w, \mathbf{c}, \mathbf{x}).$$

Now, for each word $w \in W$, it is easily verified that h has the w-antipodal property, that is,

$$\tilde{p}_w(h(\omega_w(\mathbf{x}))) = g_1(w, \mathbf{c}, \omega_w(\mathbf{x})) = -g_1(w, \mathbf{c}, x)$$

Further the zero set of h can be seen to be,

$$h^{-1}(\mathbf{0}) = \{\omega_w(S^1 \times \mathbf{c}_1) | w \in W\}, \tag{2}$$

where $\mathbf{c}_1 = (p_1(\mathbf{c}), p_{11}(\mathbf{c}), \ldots, p_{12\ldots2}(\mathbf{c}))$.

Consider now the smooth homotopy $F_m : (S^1)^{2^n} \times [0, 1] \to \mathbb{R}^{2^n-1}$ between h and δ,

$$F_m(x_1, x_2, \ldots, x_n, t) = (1 - \lambda(t))\delta_m(x_1, x_2, \ldots, x_n) + \lambda(t)h(x_1, \ldots, x_n),$$

where $\lambda : [0,1] \to [0,1]$ is a smooth monotonically increasing function such that $\lambda(u) = 0$ for $0 \leq u < \epsilon$ and $\lambda(u) = 1$ for $1 - \epsilon < u \leq 1$ and is strictly increasing in $[\epsilon, 1 - \epsilon]$. By perturbing δ_m, if necessary, while also making sure that it satisfies the conditions (a),(b) and (c) above, $\mathbf{0}$ can be made a regular value of F_m. So, ${F_m}^{-1}(\mathbf{0})$ is a compact smooth 2-manifold M , whose boundary can only be at $(S^1)^{2^n} \times 0$ and at $(S^1)^{2^n} \times 1$ (we will refer to these as the boundaries of M at $t = 0$ and $t = 1$ resp.). Observe that $M \cap (t = 0) = (h^{-1}(\mathbf{0}), 0)$ and $M \cap (t = 1) = (\delta_m^{-1}(\mathbf{0}), 1)$. Among the components of $M \cap (t = 0)$, we will be interested in the component M_U that contains the cycle $U \times 0$. We will show that the boundary of M_U at $t = 1$ has a loop, Λ_m on which the projection map $p_0 : (S^1)^{2^n}|_{\Lambda_m} \to S^1$ defined by $p_0(x_0, \ldots,) = x_0$ is not null-homotopic. There are four broad steps in proving this: first, we prove that the boundary of M_U at $t = 0$ is exactly $U \times 0$. Using this, we will show that M_U defines a framed-cobordism between its boundaries at $t = 0$ and $t = 1$. We will then show that a homotopy exists between functions whose 1-section define the boundaries of M_U at $t = 0$ and $t = 1$. This homotopy will be used to show that the mod 2 degree of the projection map p_0 restricted to $M_U \cap (t = 1)$ is non-zero, thus proving that the projection map p_0 on $\partial M_U \cap (t = 1)$ is *not* null-homotopic.

1.0.1. By definition,

$$M \cap (t = 0) = h^{-1}(\mathbf{0}) \times 0 = \{U_w \times 0 | U_w = \omega_w(S^1 \times \omega_w(\mathbf{c}_1)) \text{ and } w \in W\}$$

Let $U_w = S^1 \times \omega_w(\mathbf{c}_1)$ be one of circles in $h^{-1}(\mathbf{0})$ disjoint from U. We now show that $(U_w \times 0) \cap M_U = \emptyset$. Being a boundary of a manifold, ∂M_U consists of cycles only and so it either contains the entire cycle $U_w \times 0$ or it does not intersect $U_w \times 0$ at all. Assume for the sake of contradiction that $U_w \times 0 \subset M_U$. In what follows, in a slight abuse of notation we write map $(\mathbf{x}, t) \mapsto \delta_m(\mathbf{x})$ also as δ_m. Since, M_U is path-connected, we may assume that there is path $\gamma : [0,1] \to M_U$ with $\gamma(0) = (x_0, \mathbf{c}_1, 0)$ and $\gamma(1) = (x_0, \omega_w(\mathbf{c}_1), 0)$ for a generic $x_0 \in S^1$. Further, we may assume that $x_0 \in S^1$ is such that $\tilde{p}_w(\delta_m(\gamma(0))) \neq 0$, then as $\tilde{p}_w \circ \delta_m(\gamma(0)) = -\tilde{p}_w \circ \delta_m(\gamma(1))$, we expect $\tilde{p}_w \circ \delta_m$ to change signs an odd number of times along γ. So to arrive at a contradiction, we need to show that γ meets Σ an even number of times. To this end, we make some convenient assumptions about γ and Σ.

1. Perturbing δ_m if necessary, we may assume that Σ is a compact 1-manifold-with-boundary, consisting of finite number of line segments.
2. We may assume that γ is smooth since M_U is smooth and write
$$\gamma(s) = (x_1^\gamma(s), x_2^\gamma(s), \ldots, x_{2^n}^\gamma(s), t^\gamma(s))$$
where $x_i^\gamma : [0,1] \to S^1$ $(i = 1, \ldots, 2^n)$ and $t^\gamma : [0,1] \to \mathbb{R}$ are smooth functions.
3. Existence of collar neighborhoods $U \times [0, \epsilon)$ and $U' \times [0, \epsilon)$ in M_U implies that we may also assume that the derivatives of the first coordinate of γ, $(x_1^\gamma)^{(i)}(0)$ for $i = 1, 2, \ldots$ are all zero.
4. Further, we may assume that for each $s \in [0,1]$, $t_\gamma(s) < 1 - \epsilon$, that is, γ stays away from the collar neighborhood in M_U at $t = 1$ where $\delta_m = 0$.
5. Perturbing γ if necessary, we may also assume that Σ intersects γ in only a finite set of points, and that there are no tangential intersections.
6. Also, as Σ consists of finite set of line segments, we may assume that the points $\gamma(0)$ and $\gamma(1)$ on $U \times 0$ and $U' \times 0$ respectively are not in Σ.

With these assumptions, we will show that $p_w \circ \delta_m$ changes sign an even number of times between $\gamma(0)$ and $\gamma(1)$.

Now, the map $\tilde{p}_w \circ \delta_m$ changes sign only at points in

$$\Sigma = \{(\mathbf{x}, t) \in M | (\tilde{p}_w \circ \delta_m)(\mathbf{x}) = 0, \mathbf{x} \in (S^1)^{2^n}, t \in [0,1], w \in W\}$$

By assumption, 0 is a regular value of $p_w \circ \delta_m$ so $p_w \circ \delta_m(x, \mathbf{c}_1) = 0$ has a finite and in fact, even number of solutions for x, say, $\{w_1, w_2, \ldots, w_{2l}\}$.

The map $x_1^\gamma : [0,1] \to S^1$, which has $x_1^\gamma(0) = x_1^\gamma(1)$ and has all the derivatives vanishing at 0 and 1, defines a smooth map $\tilde{x}_1^\gamma : S^1 \to S^1$ with $\tilde{x}_1^\gamma(e^{i2\pi s}) = x_1^\gamma(s)$ for $s \in [0,1]$. By theorem on pg 24 of Milnor, [1], the value: $\#\tilde{x}_\gamma^{-1}(c) \mod (2)$ (here # as in Milnor's text indicates cardinality) is the same for each regular value c of $\tilde{x}_1^\gamma$, and so

$$\#(\tilde{x}_1^\gamma)^{-1}(\{w_1, w_2, \ldots, w_{2n}\}) \mod 2 = 2n\#(\tilde{x}_1^\gamma)^{-1}(w_1) \mod 2 = 0 \mod 2$$

The path γ hence intersects Σ even number of times and $p_w \circ \delta_m$ changes sign even number of times along γ, contradicting the fact that $p_w \circ \delta_n(\gamma(0)$ and $p_w \circ \delta_m(\gamma(1)$ have opposite signs. It follows that such a path γ between the points in U and U_w is not possible. Hence $M_U \cap (t = 0) = U \times 0$.

1.0.2. Let $V = \partial M_U \cap (t = 1)$. We show now that, V is the $\mathbf{1}$-section of a continuous map $\beta : (S^1)^{2^n} \to (S^1)^{2^n-1}$ with $p_w(\beta(x, \mathbf{y})) = y_w e^{i\phi_w(x,y)}$ for some continuous map $\phi : S^1 \times S^1 \to \mathbb{R}$.

The maps h and δ_m being smoothly homotopic, define a framed cobordism between framed manifolds $h^{-1}(0)$ and $\delta_m^{-1}(0)$ (see Lemma 3 of Milnor,[1] pg. 45). Within the sub-manifold M_U of M we then have a framed-cobordism between framed sub-manifolds $U = \partial M_U \cap (t = 0)$ (1.0.1) and $V = \partial M_U \cap (t = 1)$, the framing at U and V being the restriction of frames on the framed manifold $F_m^{-1}(0)$. Let υ denote induced frame on U. We would like to treat U as Pontryagin manifold while retaining the frame υ. The natural candiate is the projection map $p_2 : (S^1)^{2^n} \to (S^1)^{2^n-1}$ defined by $p_2(x, \mathbf{y}) = \mathbf{y}$, for which $U = p_2^{-1}(\mathbf{c}_1)$. We claim that the frame υ is also induced by p_2, that is $\upsilon = p_2^*(\mathfrak{w})$ ($*$ notation as in Milnor) for some suitable set of $2^n - 1$ basis vectors $\mathfrak{w}(\mathbf{y})$ of in tangent space $T_{\mathbf{y}}((S^1)^{2^n-1})$. This is true since $2^n - 1$ basis vectors $\upsilon(x, \mathbf{y})$ are mapped by the differential $Dp_2(x, \mathbf{y})$ to a independent set of vectors in the tangent space at $\mathbf{y}$ of $(S^1)^{2^n-1}$. The framed cobordism between the boundaries of M_U implies by Theorem C, pg 44 of Milnor [1] that V is also a Pontryagin manifold for some smooth mapping $\beta : (S^1)^{2^n} \to (S^1)^{2^n-1}$. That is $V = \beta^{-1}(\mathbf{1})$ with a frame induced a choice of tangent vectors on $(S^1)^{2^n-1}$ at $\mathbf{1}$. So by the homotopy-cobordism correspondence theorem (Theorem B of Milnor [1] pg 43) we have a homotopy between $\beta : (S^1)^{2^n} \to (S^1)^{2^n-1}$ and $p_2 : (S^1)^{2^n} \to (S^1)^{2^n-1}$. This then implies for each $w \in W$, $p_w \circ \beta \cong p_w \circ p_2$. So by Thm 5.1, pg 561, of [2]), we have,

$$p_w \circ \beta(x, \mathbf{y}) = p_w \circ p_2(x, \mathbf{y}) e^{i\phi_w(x,\mathbf{y})}$$

for some real-valued continuous function $\phi_w : S^1 \times S^1 \to \mathbb{R}$.

1.0.3. We next show that the mod 2 degree of the projection map $(x, \mathbf{y}) \to x$ on $(S^1)^{2^n}$ defined on V, is 1. Since $V = \beta^{-1}()$, the mod 2 degree of this map, can be found by determining the parity of the number of solutions for y in the following system of equations, for a generic $x_0 \in S^1$.

$$y_w e^{i\phi(x_0,\mathbf{y})} = 1 \quad \text{for each} \quad w \in W \tag{3}$$

Notice this may regarded as a problem of finding the mod 2 degree of the map $\beta' : (S^1)^{2^n-1} \to (S^1)^{2^n-1}$ defined by:

$$p_w(\beta'(\mathbf{y})) = y_w e^{i\phi_w(x_0,\mathbf{y})}$$

This map being homotopic to the identity map has an odd degree. Thus the projection map $(x, \mathbf{y}) \to x$ defined on V also has odd degree.

$\square$

Lemma 1. *Let P be a real valued function defined on the connected closed subset Λ on the torus $(S^1)^k$ such that projection $p_1 : S^1 \times (S^1)^{k-1} \to S^1$ defined by $p_1(x, \mathbf{y}) = x$ is not null-homotopic. Then there exists two points $(x, \mathbf{y_1})$ and $(-x, \mathbf{y_2})$ in Λ such that $P(x, \mathbf{y_1}) = P(-x, \mathbf{y_2})$.*

Proof. Let $g : \Lambda \to S^1 \times \mathbb{R}$ be the map $g(x, \mathbf{y}) = (x, P(x, \mathbf{y}))$. Define $L = g(\Lambda)$. Since the map $p_1 : \Lambda \to S^1$ is not null-homotopic, the projection map $\tilde{p}_1 : L \to S^1$ defined by $(x, w) \to x$ is also not null-homotopic, for if $\tilde{p}_1 \cong \mathbf{1}$ ($\mathbf{1}$ be a constant function), then $p_1 = \tilde{p}_1 \circ g \cong \mathbf{1} \circ g \cong \mathbf{1}$.

Now let ϕ be the standard homeomorphism from the cylinder $S^1 \times \mathbb{R}$ to the punctured complex plane $\mathbb{C} \backslash \{0\}$, defined by,

$$\phi(e^{i\theta}, t) = (e^t e^{i\theta})$$

Claim: The origin 0 is in a bounded component of $\mathbb{C} \backslash \phi(L)$.

Consider the map $\varphi : \mathbb{C} \backslash \{0\} \to S^1$ defined by $z \to z/|z|$. When restricted to $\phi(L)$ this map is not null-homotopic, since $\tilde{p}_1 = \varphi \circ \phi$ is not null-homotopic on L. If 0 is not in any of the bounded components of $\mathbb{C} \backslash \phi(L)$, then there exists a unbounded polygonal arc $E \subset \mathbb{C} \backslash \phi(L)$ with one end as 0, but $\mathbb{C} \backslash E$ is contractible, so the map $\varphi|_{\mathbb{C} \backslash E} \cong 1$ and so its restriction to $\phi(L)$, $\varphi|_{\phi(L)}$ is also null-homotopic, contradicting the fact φ is not null-homotopic on $\phi(L)$. Hence the origin must be one of the bounded components of $\mathbb{C} \backslash \phi(L)$.

Claim: $\phi(L)$ contains a pair of antipodal points. The loop L and therefore $\phi(L)$ are both compact, and by the previous claim, the origin 0 is in a bounded component of $\mathbb{C} \backslash \phi(L)$, so we may apply Lemma **??** to conclude that $\phi(L)$ contains a pair of antipodal points $z = rw$ and $-z = -rw$, where $r = |z|$.

The points z and $-z$ correspond to the points $\phi^{-1}(z) = (w, \log r)$ and $\phi^{-1}(-z) = (-w, \log r)$ respectively in L. But every point in L is of the form $(x, P(x, \mathbf{y}))$ and so conclude that there are points $(x, P(x, \mathbf{y}_1))$ and $(-x, P(-x, \mathbf{y}_2))$ in L such that $P(x, \mathbf{y}_1) = P(-x, \mathbf{y}_2)$. □

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